CLASSIFICATION OF THE INVARIANT SOLUTIONS TO THE EQUATIONS FOR THE TWO-DIMENSIONAL TRANSIENT-STATE FLOW OF A GAS

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Here one considers all invariant solutions to the system of equations for two-dimensional gas dynamics:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\operatorname{grad} p = 0,$$
$$\frac{\partial \rho}{\partial t} + (\mathbf{v}\cdot\operatorname{grad}\rho) + \rho \operatorname{div}\mathbf{v} = 0$$
$$\frac{\partial p}{\partial t} + (\mathbf{v}\cdot\operatorname{grad} p) + A \operatorname{div}\mathbf{v} = 0,$$
$$\left(A = A(p,\rho) \equiv -\rho\frac{\partial S/\partial\rho}{\partial S/\partial p}\right). \qquad (1)$$

Here p is pressure, ρ is density, S is entropy, and $\mathbf{v} = \mathbf{v}(x, y)$ is the velocity vector, whose components are u and v; it is assumed that $\partial S/\partial P \neq 0$. Two cases will be considered.

Case A: $A(p, \rho)$ an arbitrary function.

Case B: $A = \gamma p$, a polytropic gas with γ = constant.

The principal group of transformations allowed by (1) has been given [1], and for case A the basis of the corresponding Lie algebra consists of the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_4 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad X_5 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v},$$
$$X_2 = \frac{\partial}{\partial x}, \quad X_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$
$$X_3 = \frac{\partial}{\partial y}, \quad X_7 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \qquad (2)$$

while in case B we add to these the operators

$$X_8 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + 2\rho\frac{\partial}{\partial \rho},$$
$$X_9 = \rho\frac{\partial}{\partial \rho} + p\frac{\partial}{\partial p}. \qquad (3)$$

Table 1

1	$X_1 + \delta X_9 \quad (\delta = 0, 1)$	8	$X_3 + X_8 + \delta X_9$
2	$X_2 + \delta X_9 \quad (\delta = 0, 1)$	9	$X_4 + X_6 + \delta X_9$
3	$X_1 + X_4 + \delta X_9 \ (\delta = 0, 1)$	10	$\alpha X_6 + \beta X_7 + X_8 + \delta X_9$
4	$X_2 + X_5 + \delta X_9 \ (\delta = 0, 1)$	11	$X_1 - X_6 + \alpha X_7 + X_8 + \delta X_9$
5	$X_6 + \alpha X_7 + \delta X_9$	12	X_9
6	$X_7 + \delta X_9$	13	$X_5 + \delta X_9 \quad (\delta = 0, 1)$
7	$X_1 + X_7 + \delta X_9$		

For $\gamma = 2$ we add to (2) and (3) the operator

$$X_{10} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + (x - tu)\frac{\partial}{\partial u} + $$
$$+ (y - tv)\frac{\partial}{\partial v} - 4tp\frac{\partial}{\partial p} - 2t\rho\frac{\partial}{\partial \rho}. \qquad (4)$$

The basic group for case A is denoted by G_7, while for case B it is denoted by G_9 for arbitrary γ and by G_{10} for $\gamma = 2$.

Table 1 gives the optimal system of one-parameter subgroups of group G_9.

The optimal system of one-parameter subgroups of group G_7 consists of operators 1–7 of Table 1 for $\delta = 0$ with operator $X_4 + \alpha X_6$, while the same for group G_{10} consists of operators 1–12 of Table 1 and the operators

$$14.\ X_1 + X_2 + X_7 + \delta X_9 + X_{10},$$
$$15.\ X_1 + \alpha X_6 + \beta X_7 + \delta X_9 + X_{10}. \qquad (5)$$

Table 2

1	$X_1, \quad X_2$	6	$\alpha X_4 + X_6$	10	$X_1 + X_4 + \alpha X_5$
2	X_6	7	X_7, X_6	11	$\alpha X_4 + \beta X_5 + X_6$
3	$\alpha X_6 + X_7$	8	X_2, X_3	12	X_4
4	$X_5, \quad X_4$	9	$X_3 + X_4$	13	$X_1 + X_5$
5	$X_2 + \alpha X_3$			14	$X_2 + X_5, \alpha X_2 + \beta X_3 + X_4$

The α and β of the last operator satisfy $0 \leq \alpha < 2$ and $\beta \geq 0$ for $\alpha = 0$.

Table 2 gives the optimal system of two-parameter subgroups of group G_7; Table 4 does the same for group G_9, and subgroups 1–40 of Tables 3 and 4 do the same for group G_{10}.

The form of the invariant solutions of rank unity is as follows. These solutions are derived from the two-parameter subgroups. U, V, P, and R are dependent on a single argument λ, whose expressions in terms of t, x, and y vary with the subgroup and are given below. The necessary condition for an invariant solution is not obeyed for subgroups in which operator X_9 is one of the forming elements; moreover, the X_9 term in all subgroups affects only p and ρ, and this effect is easily allowed for, so X_9 will not be considered. Also, I do not consider subgroups in which as one of the forming elements we have X_1, X_2, X_3, X_4, or X_5, since these give the stationary and one-dimensional case. For instance, the invariant solution for $H = \langle X_5 \rangle$ takes the form

$$u = U(t, x), \quad v = \frac{y}{t} + V(t, x),$$
$$p = \frac{1}{t}P(t, x), \quad \rho = \frac{1}{t}R(t, x).$$

Then the H of system (1) is

$$V_t + UV_x + t^{-1}V = 0, \quad U_t + UU_x + R^{-1}P_x = 0,$$
$$R_t + UR_x + RU_x = 0, \quad P_t + UP_x + A'U_x = 0$$
$$\left(A' = -R\frac{\partial S/\partial R}{\partial S/\partial R}\right).$$

We have a system of equations for one-dimensional motion for U, P, and R, while V is found from

$$V_t + UV_x + t^{-1}V = 0$$

with a known function U(t, x), so we have to deal with the solution of equations for one-dimensional motion.

For the subgroups of Table 2 we get invariant solutions of the form

$$7.\ u_r = U, \quad u_\varphi = V, \quad p = P, \quad \rho = R, \quad \lambda = r/t.$$

Table 3

1	$X_9,$	$X_1 + X_2 + X_7 + X_{10}$	
2		$X_1 + \alpha X_6 + \beta X_7 + X_{10}$	$(0 \leqslant \alpha < 2)$
3	$X_2 + X_5,$	$X_1 + \alpha X_6 - X_7 - \alpha X_8 + \beta X_9 + X_{10}$	$(\alpha \neq 0)$
4		$X_1 + \alpha X_2 + \beta X_3 + \delta X_9 + X_{10} - X_7$	
5	$X_2 + X_5 + X_9,$	$X_1 + \alpha X_2 + \beta X_3 - X_7 + X_{10}$	
6	$X_7 + \delta X_9,$	$X_1 + \alpha X_6 + \beta X_9 + X_{10}$	$(0 \leqslant \alpha < 2)$
7	$X_6 + \varepsilon X_7 - X_8 + \delta X_9,$	$X_1 + \alpha X_6 + \beta X_7 + \varkappa X_9 + X_{10}$	$(0 \leqslant \alpha < 2)$

Here r and φ are polar coordinates in the (x, y) plane, while u_r and u_φ are the projections of the velocity on the axes of the polar coordinates.

$$14.\ u = \frac{tx - y}{t^2 + \alpha t - \beta} + U, \quad v = \frac{(\alpha + t) y - \beta x}{t^2 + \alpha t - \beta} + V, \quad p = P,$$

$$\rho = R, \quad \lambda = t.$$

In Table 4 we need consider only subgroups 9–14:

$$9.\ u_r = r^{-1/\beta} U, \quad u_\varphi = r^{-1/\beta} V, \quad p = P, \ \rho = r^{2/\beta} R,$$

$$\lambda = t r^{-(\beta+1)/\beta} \ \text{for} \ \beta \neq 0,$$

$$u_r = t^{-1} U, \quad u_\varphi = t^{-1} V, \quad p = P, \ \rho = t^2 R,$$

$$\lambda = r \quad \text{for} \quad \beta = 0,$$

$$10.\ u = t + \sqrt{y}\, U, \quad v = \sqrt{y}\, V, \quad p = P, \quad \rho = y^{-1} R,$$

$$\lambda = 1/2 \quad t^2 y^{-1} - x y^{-1},$$

$$11.\ u = (tx - y)\, U, \quad v = x + (tx - y) V, \quad p = P,$$

$$\rho = (tx - y)^{-2} R; \ \lambda = t,$$

$$12.\ u_r = r\, U, \quad u_\varphi = rV, \quad p = P, \quad \rho = r^{-2} R, \quad \lambda = re^t,$$

$$13.\ u_r = r\, U, \quad u_\varphi = rV, \quad p = P, \quad \rho = r^{-2} R,$$

$$\lambda = t + \varphi + \beta \ \ln \ r,$$

$$14.\ u_r = r^{\alpha/\delta} e^{\varphi/\delta} U, \quad u_\varphi = r^{\alpha/\delta} e^{\varphi/\delta} V, \quad p = P, \quad \rho = t^2 r^{-2} R,$$

$$\lambda = r^{\alpha - \delta} t^\delta e^\varphi \quad \text{for} \quad \delta^{-2} \neq 0,$$

$$u_r = r t^{-1} U, \quad u_\varphi = r t^{-1} V, \quad p = P, \quad \rho = t^2 r^{-2} R,$$

$$\lambda = \varphi + \alpha \ \ln \ r \ \text{for} \ \delta = 0.$$

For Table 3 we have

$$3.\ u = \frac{e^{\theta(t)}}{1 + t^2} (U - tV + \lambda t^2),$$

$$v = x + \frac{e^{\theta(t)}}{1 + t^2} [V + tU - \lambda t (t^2 + 2)],$$

$$p = \frac{P}{(1 + t^2)^2}, \quad \rho = \frac{e^{-2\theta(t)} R}{(1 + t^2)},$$

$$\lambda = \frac{(tx - y)\, e^{-\theta(t)}}{1 + t^2},$$

$$4.\ u = \frac{U - tV + \lambda t^2 + {}^1/_2 \beta t}{1 + t^2} - \frac{\beta \theta(t)}{2\alpha},$$

$$\rho = \frac{R}{1 + t^2}, \quad p = \frac{P}{(1 + t^2)^2},$$

$$v = x + \frac{V + tU - \lambda t (t^2 + 2) + {}^1/_2 \beta t^2}{1 + t^2} + \frac{t \beta \theta(t)}{2\alpha},$$

$$\lambda = \frac{tx - y + {}^1/_2 \alpha + {}^1/_2 \beta t}{1 + t^2} + \frac{\beta}{2\alpha} \theta(t),$$

$$\theta(t) = \alpha \ \text{arc tg}.$$

6. To avoid complicating the formulas we consider the case $\alpha = \beta = 0$:

$$u_r = \frac{rt}{1 + t^2} + \frac{U}{r}, \quad u_\varphi = \frac{V}{r}, \quad p = \frac{P}{(1 + t^2)^2}, \quad \rho = \frac{R}{1 + t^2},$$

$$\lambda = \frac{r}{\sqrt{1 + t^2}}.$$

7. Here also we assume $\alpha = \beta = 0$:

$$u_r = \frac{rt}{1 + t^2} + \frac{rU}{1 + t^2}, \quad u_\varphi = \frac{rV}{1 + t^2}, \quad p = \frac{P}{(1 + t^2)^2},$$

$$\rho = \frac{R}{r^2}, \quad \lambda = \varphi + \delta \ln \frac{r}{\sqrt{1 + t^2}}.$$

Table 4

1	$X_1 + X_9, \ \alpha X_1 + X_7$	26	$X_7 + \alpha X_8 + \beta X_9$
2	$\alpha X_1 + X_6 + \beta X_7 - X_8$	27	$X_6 + \alpha X_7 + \beta X_8 + \delta X_9$
3	$X_2 + X_9, \ X_3 + X_4$	28	X_9
4	$X_1 + \alpha X_4 + \beta X_5$		$X_2, \ X_1 + \alpha X_4 + \beta X_5 + \delta X_9$
5	$\alpha X_2 + \beta X_4 + X_5 \ (\alpha = 0, \ 1)$	29	$(\alpha = 0, \ 1)$
6	$\alpha X_2 + \beta X_3 + X_8$	30	$(\delta = 0, \ 1)$
7	$X_2 + X_5, \ \alpha X_2 + X_4 + \beta X_3 + \delta X_9$	31	$\alpha X_4 + X_5 + \beta X_9 \ (\beta = 0, \ 1)$
	$(\delta = 0, \ 1)$		$X_3 + X_4 + \alpha X_9 \quad (\alpha = 0, \ 1)$
8	$X_6 + \beta X_9$	32	$X_3 + \alpha X_9 \quad (\alpha = 0, \ 1)$
9	$X_7 + \alpha X_9, \quad \beta X_6 + X_8 + \delta X_9$	33	$X_4 + \alpha X_9 \quad (\alpha = 0, \ 1)$
10	$X_1 + X_4, \ -2X_6 + X_8 + \alpha X_9$	34	$X_4 + \alpha X_5 + X_6 + \beta X_9$
11	$X_2 + X_5, \ -X_6 + X_8 + \alpha X_9$	35	$X_5 + X_6 + \alpha X_9$
12	$X_7 + dX_9, \ X_1 - X_6 + X_8 + \beta X_9$	36	$X_6 + \alpha X_9$
13	$X_1 + X_7 + \alpha X_9, \ \beta X_1 - X_6 +$	37	$X_3 + X_8 + \alpha X_9$
	$+ X_8 + \delta X_9$	38	$\alpha X_6 + X_8 + \beta X_9$
14	$X_6 + \alpha X_7 + \beta X_9, \ \delta X_7 + X_8 + \varepsilon X_9$	39	$X_1 - X_6 + X_8 + \alpha X_9$
15	$X_9, \quad X_1 + X_4$	40	X_9
16	$X_2 + X_5$	41	$X_5, \ X_3 + X_9$
17	$X_6 + \alpha X_7$	42	$X_2 + \alpha\beta X_3 + X_4 + \beta X_9 \ (\beta = 0, \ 1)$
18	X_7	43	$X_3 + X_4 + X_9$
19	$X_1 + X_7$	44	$X_4 + \alpha X_9 \quad (\alpha = 0, \ 1)$
20	$X_2 + X_8$	45	$X_3 + X_8 + \alpha X_9$
21	$X_4 + X_6$	46	$\alpha X_4 + X_6 + \beta X_9 \ (\alpha = 0, \ 1)$
22	$\alpha X_6 + \beta X_7 + X_8$	47	$X_2 + \alpha X_3 + X_8 + \beta X_9$
23	$X_1 - X_6 + \alpha X_7 + X_8$	48	$\alpha X_6 + X_8 + \beta X_9$
24	$X_1, \quad X_8 + \alpha X_9$	49	X_9
25	$X_2 + X_8 + \alpha X_9$	50	$X_5 + X_9, \ \alpha X_4 + \beta X_5 + X_6$

The form of the invariant solutions of second rank is also found without difficulty.

As an example we consider the invariant solution corresponding to the subgroup $H = \langle X_7, X_1 + X_{10}\rangle$. This has the form

$$u_r = \frac{rt}{1+t^2} + \frac{U}{r}, \quad u_\varphi = \frac{V}{r}, \quad \rho = \frac{R}{1+t^2}, \quad p = \frac{P}{(1+t^2)^2},$$

$$\lambda = \frac{r}{\sqrt{1+t^2}}. \qquad (6)$$

In this case $\gamma = 2$, and the equations of gas dynamics may be interpreted as those of shallow water. Without loss of generality we may consider the density of the water and the acceleration due to gravity as unity; then $p = \rho^2/2$, and ρ is the height of the water above the even base. Consider the motion of the water over a dry base. We substitute (6) into (1), as written in polar coordinates, and find one of the solutions as

$$U = 0, \quad V = 0, \quad R = {}^1/_2\,(a^2 - \lambda^2), \quad \lambda \leqslant a, \quad a = \text{const}.$$

This describes the flow of a hill of water; consider now the motion over a dry even place of a mass of water that at $t = 0$ has the form $\rho = {}^1/_2\,(a^2 - r^2)$, $r \leqslant a$ and that is at rest. The solution is

$$u_r = \frac{rt}{1+t^2}, \quad u_\varphi = 0, \quad \rho = \frac{1}{2(1+t^2)}\left(a^2 - \frac{r^2}{1+t^2}\right).$$

The boundary ($\rho = 0$) moves in accordance with $r = a\sqrt{1+t^2}$, while the height of the vertex ($r = 0$) decreases in accordance with the law $[a^2/(1 + t^2)]/2$. The velocity remains bounded, $u_r < a$. In this solution the velocity is a linear function of the coordinates, so it is one of the class of solutions found previously by Ovsyannikov [2].

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